THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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LONDON:

GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN, AND BOMBAY.

VOL. III. JULY, 1904. No. 46.

NOTICE.

THIRD INTERNATIONAL CONGRESS OF MATHEMATICIANS AT HEIDELBERG, AUGUST, 1904.

MEMBERS of the Mathematical Association are invited to be present at the Third International Congress of Mathematicians, which will take place at Heidelberg from Monday, August 8th, to Saturday, August 13th, 1904. The proceedings will commence with a reception at 8 P.M. on August 8th; there will be three general meetings, and also meetings of six sections, viz., (1) Arithmetic and Algebra, (2) Analysis, (3) Geometry, (4) Applied Mathematics, (5) History of Mathematics, (6) Pedagogy.

The centenary of the birth of C. G. J. Jacobi will be celebrated during the Congress.

There will be an exhibition of Mathematical Models (historical and modern), and an exhibition of the more important mathematical publications during the past ten years, to which contributions on loan are invited. Intending contributors to the exhibition of literature are requested to correspond with Dr. A. Gutzmer, Professor an der Universität Jena. Social meetings, a banquet, and an illumination of the Castle will also take place.

The price of a ticket for the Congress is 20 marks—additional tickets may be obtained by members of the Congress for their own party at 10 marks each person, which are available for the general meetings and festivities.

Further information concerning the Congress may be had from Professor A. Krazer, Karlsruhe i. B. Westendstrasse, 57.

REPORT OF THE M.A. COMMITTEE ON THE TEACHING OF ELEMENTARY MECHANICS.

THE Committee make the following suggestions:—

A. Preliminary Experimental Work.

1. That a simple course of experiments would be useful with a view to illustrating
   - (α) Composition and resolution of forces.
   - (β) The principle of the lever.
   - (γ) Friction.

2. The experiments should, if possible, aim at discovery, and be quantitative—mere verification is less useful.

B. General Remarks on Examples and Methods.

3. Examples should at first, as a rule, be numerical, and any of a general character should conclude with a numerical application.
4. Examples should, as far as possible, be of a practical nature.
5. A specially instructive class of example consists in compiling a table or drawing a graph to show the effect on a result of variation in a certain datum.
6. Stress should be laid on the great importance of checking results by an independent method; in particular, questions should often be worked out both graphically and by calculation.
7. Prominence should be given to geometrical methods.
8. Pupils should always be required to specify the body whose equilibrium or motion is being considered, and to indicate the complete system of forces acting on the body before writing down any equations.
9. Simplifying assumptions, such as that friction, stiffness of ropes, weights of certain bodies, etc., are being disregarded in any particular question, cannot be too explicitly stated.
10. Fancy names for technical terms are to be avoided.

C. Statics.

11. As the basis of the subject the parallelogram of forces should be assumed as an experimental result.
12. This should be immediately followed by problems on the equilibrium of bodies acted on by three concurrent forces, to be solved graphically.
13. The calculation methods should then follow the graphic methods, and should be applied to numerical cases in which four figure tables should be used, the angles 30°, 45°, 60° playing a very small part.
14. The Committee suggest that any of the following methods of dealing with the problem of parallel forces should be regarded as admissible:
    (α) By taking moments, assuming that a clear idea of the meaning of a moment and its mode of measurement have been obtained by experiment.
    (β) By the use of the funicular polygon, or by the equivalent geometrical device which deduces parallel from nonparallel forces.
    (γ) By defining a couple and proving from the parallelogram of forces that it is measured by its moment.
15. Both the graphical and analytical methods of determining the resultant of any number of coplanar forces should be taught, and numerical examples should be given.
16. It should be pointed out that all composition of forces assumes the existence of a rigid body to which the forces are applied, and that, failing the existence of such a body, composition of forces is unlawful and indeed unmeaning.
17. The impression, that the weight of a body is in reality a single force acting at its centre of gravity, should be guarded against. This and other cases where rigidity is assumed should be impressed on the beginner by contrasting rigid bodies with bodies which are not rigid.
18. *Machines.*—(α) The phrase "Mechanical Advantage" may preferably be replaced by the term "Force-ratio," viz., the ratio of the resistance to the effort when the effort is just overcoming the resistance.

(β) The consideration of work should be an essential part of the discussion of machines, and attention should at an early stage be given to "Velocity-ratio" and "Efficiency."

(γ) Systems of pulleys should not be referred to by numbers.

19. When the equilibrium of two or more connected bodies or parts of a single body is considered, the principle of "Separate Equilibrium" should be distinctly enunciated.
20. It should be clearly pointed out that all the results of statics apply to cases of uniform motion.

D. Kinetics (commonly called Dynamics).*

21. *Velocity.*—The meaning of the phrase "velocity at a point" should be carefully brought out by means of the idea of "average velocity." Average velocity should be defined as "total distance / total time," and should not be assumed to be identical with the arithmetic mean of the initial and final velocities, or with the velocity at half-time, or with the velocity at half-way.
22. Much stress should be laid on Newton's first law and many examples given.
23. *Angular velocity* should receive attention, as, in connection with it, a great variety of interesting examples arise, *e.g.* on the gearing of wheels.
24. *Acceleration.*—The velocity at any time should be represented graphically. This method should be used to illustrate the idea of acceleration, and the formulæ for uniformly accelerated motion should be obtained from the fact that the graph is in this case a straight line.
25. The formulæ for uniform acceleration having been proved as above, the fact that "the average velocity = the velocity at the middle instant" should be frequently employed in solving problems connected with such motion.
26. It should be expressly stated that all velocities with which we deal are necessarily relative to some base.
27. In explaining the parallelogram of velocities, it should be stated that we are proposing to determine the velocity of a point A relative to a base C from a knowledge of the velocity of A relative to a base B and of the velocity of B relative to the base C.
28. It should be permissible to treat elementary problems on the accelerations produced by forces by simple proportion

$$\left(\frac{\text{force acting}}{\text{acceleration produced}} = \frac{\text{weight}}{g}\right)$$

using the fact that a body's own weight produces acceleration g; and it should be allowable to postpone the consideration of mass until such problems have been discussed.
29. The poundal, when employed, should be used as an auxiliary unit, final results being reduced to gravitational units.
30. Atwood's machine should be regarded as a means of illustrating the laws of motion, and not as a practical method of finding g. The unsound method [mass moved $= m + m'$; moving force $= (m - m')g$] is to be condemned.
31. With the idea of preventing the notion that acceleration is always uniform, and having regard to the importance in physics of simple harmonic motion, it is advisable to consider such motion and the pendulum at an early stage, before the difficult parts of the work on projectiles and before oblique collisions.
32. Easy problems on the motion of a fly-wheel should form part of a course on elementary mechanics.
33. The phrase "centrifugal force" should be abandoned.
34. It should be clearly pointed out that the principle of the "Conservation of Energy" is not capable of universal application to mechanical problems,

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* The term "Dynamics" more properly connotes the whole science of force.

whereas the principle of the "Conservation of Momentum" can always be applied.

35. It should be pointed out that all the parallelogram laws are cases of a single law, that of the addition of vectors.

36. There should be no objection to illustrating the idea of a rate so as to lead up to the elementary ideas of the calculus.

E. Order of Teaching.

37. A short course of easy numerical trigonometry should precede the theoretical study of mechanics.

38. While several suggestions as to the order of (*a*) Statics, (*b*) Kinetics have been made above, the Committee does not wish to recommend that Statics should precede Kinetics, or *vice versa*.

REPORT OF THE M.A. COMMITTEE ON ADVANCED SCHOOL MATHEMATICS.

In considering this subject the Committee have taken account of the different classes of boys who study what may be termed "advanced mathematics" at school. These include candidates for army examinations, science students, engineering students, as well as boys who intend to read for mathematical schools at the universities. Those who intend to take up engineering will often be candidates for mathematical scholarships.

The Committee consider it desirable that, in order to secure the highest possible efficiency in teaching mathematics in schools, the schedules of examinations should be so framed that candidates may, so far as possible, all follow the same preliminary course.

The Committee think that a preliminary advanced course suitable for all classes of students should include:

(1) The following portions of Algebra: partial fractions, elementary manipulation with complex number, and geometrical applications thereof, the theory of equations so far as it treats of the numerical solution of equations, the notation and easy properties of determinants, the simpler tests of convergency, and the binomial, exponential and logarithmic series.

The following parts of Algebra should be excluded from the preliminary course:—The theory of numbers, probability, continued fractions (with the exception, perhaps, of simple continued fractions), the advanced theorems on inequalities, and on indeterminate equations, and summation of series.

The preliminary course should further include:

(2) An early introduction to the differential and integral calculus and a free use of the same in subsequent work. The use of differentials shall be permitted.

(3) Graphical illustrations of Demoivre's theorem. Simple work in trigonometrical series and factors.

(4) A treatment of the elementary parts of conic sections in which either the geometrical or the analytical method is used, that method being used in each particular case which is most suitable for the problem under discussion.

(5) The elementary geometry of the plane, cone, cylinder, sphere and regular solids, including practical solid geometry.

(6) An introduction to the dynamics of rotation (in two dimensions), viz., the motion of a rigid body round a fixed axis with uniformly accelerated angular velocity, together with other simple cases of the motions of rigid bodies.

The Sub-committee also wish to express their opinion of the importance of:

(7) A more intimate union between the teaching of Mathematics and

Science, whereby theoretical and practical work may be brought into relation with one another.

The above course would form an excellent preliminary course for scholarship candidates, who would then proceed to more advanced work in the various subjects so as to arrive at about the standard which is now required.

Regarding examinations for scholarships at the Universities in their twofold capacity, (1) as a means of selection, (2) as providing a guide to work at schools, there is no reason to doubt that the former function is efficiently performed by the papers as now set.

With regard to the second function, that of directing school work, if it is decided that the above preliminary course is on the right lines, it would be desirable for scholarship examiners to recognise it by :

(*a*) Setting questions involving applications of the integral calculus ;

(*b*) requiring work in theoretical solid geometry as above indicated ;

(*c*) abolishing the restriction to "methods of pure geometry," and leaving the method employed to the judgment of the candidate ;

(*d*) discouraging the cramming of book work in advanced trigonometry by not requiring candidates to reproduce long and difficult proofs (such as that needed to establish the infinite product form for the sine).

SCHOOL CERTIFICATE.

Board of Education,
Whitehall,
12*th July* 1904.

Sir,—I am directed to bring to your notice the enclosed copy of certain suggestions which have been submitted to the Board of Education by the Consultative Committee for a system of School Certificates. They have been put forward by that Committee as the result of careful and prolonged consideration after conference with representatives of the English Universities, of Associations of Teachers, and of Professional Bodies concerned with the question, and after consulting with persons specially conversant with the administration of similar examinations.

In view of the capital importance of these proposals in relation to Secondary Schools, the Board of Education are anxious to obtain an expression of the views of the Universities, and of other bodies interested, upon the important issues raised in them, and upon the solutions suggested by the Committee for consideration.

I am accordingly directed to request that you will be good enough to take such steps as may seem to you best calculated to further the objects in view, and to favour the Board not later than December 31st, 1904, or earlier should it prove possible, with the outcome of such consideration as may be accorded to the scheme.

In issuing the Report as it has been received from the Consultative Committee, it must be clearly understood that the Board refrain from the expression of any view as to the desirability or feasibility of the proposals therein contained, and that they are not committed at this stage to any action in the matter.

I am, Sir,
Your obedient servant,
Robert L. Morant,
Secretary.

To the Secretaries
of the Mathematical Association.

BOARD OF EDUCATION.

CONSULTATIVE COMMITTEE.

Proposals for a System of School Certificates.

The Consultative Committee are of opinion :—

(1) That, with the object of diminishing the multiplicity of examinations affecting secondary schools, and of providing a test of adequate general education which may be widely accepted, a general system of school certificates is desirable.

The well-known term "Leaving Certificate" has been purposely avoided because it is to some extent misleading and is not unfrequently misunderstood.

(2) That it is not desirable that examinations for such certificates should be conducted by means of papers set for the whole country from a single central organisation.

This clause must be read together with clause (6). It will be found that it is not the intention of these proposals to ignore the influence of the State in the supervision of a general system of examinations. On the other hand, it is important to prevent the evils which would almost certainly arise from the State having the sole responsibility in the matter. The desirability of bringing the examining body into closer relation with the teacher being recognised, it is obvious that in dealing with a population of more than 30,000,000 and a large number and great variety of schools, this object can only be effectively attained by the establishment of more than one examining body. The success of the Scottish and Welsh systems seems to be largely due to the limited number of schools with which they have to deal.

(3) That such examinations should be controlled by a recognised **examining body**, which should be either (1) a University, or (2) a combination of Universities, or (3) an Examination Board representative of a University or Universities and of the local authorities which are prepared to co-operate with them. It is desirable that whatever the examining body may be, teachers of schools should, where possible, be represented, and with regard to (3), that every such Board should contain a large academic element.

The proposal to form in some cases Boards representing local authorities and teachers in the schools as well as Universities may afford an opportunity for making an important new departure. Those local authorities especially which aid the schools and may perhaps pay the examination fees may be glad to be associated with a neighbouring University. There is further a growing body of public opinion in favour of associating the teachers in the schools with duties of this kind.

No general rule can be laid down requiring a school to be examined by a particular examining body. It may often be desirable that a school should be examined by the University or Board of the district in which it is situate. On the other hand a school may prefer to preserve or to create a connection with one of the Universities of Oxford, Cambridge or London. It is recognised that it would not be desirable, if it were possible, to disregard the non-local character of these Universities, or the position which their examinations occupy all over the country. The proposals are based on the assumption that it will ultimately be best for the secondary schools which are maintained or largely aided by local authorities to look to provincial examining bodies for the organisation of their examinations, and it is not improbable that local authorities may prefer their doing so; but in any case there will be a period of transition during which the new system and the existing University examinations will run side by side for all classes of schools, and the higher secondary schools will doubtless always retain complete liberty in the choice of their examining body.

(4) That recognition of these examining bodies should mean recognition by the Board of Education, acting on the advice of the Consultative Committee.

(5) That the following conditions should be required from schools which present candidates for school certificates :—

(*a*) Periodical Inspection. Whether this inspection be conducted by officers of the Board of Education, or by a University or other

organisation recognised under Section 3 of the Board of Education Act, 1899, the report of the inspection should be communicated to the examining body.

(*b*) The communication of the course of studies pursued in the school to the examining body.

That an examining body should be at liberty to decline to examine a school if the result of the inspection has not been, in their opinion, satisfactory; or if the course of studies is such as they are not able to approve.

It is considered that in this connection inspection and examination should be treated as complementary one to the other. Inspection is required, in the first place to enable the examining body to judge whether the school is fitted to be admitted to the benefits of the system; but it is also required to enable the examiners to understand the aims and characters of the different schools, and so, on the one hand, to prevent the examination from becoming mechanical and rigid, and on the other to check any tendency in the school to direct its efforts too exclusively to success in the examination.

(6) That a **Central Board** should be established for England (excluding, for the present, Wales and Monmouth), consisting of representatives from the Board of Education and from the different examining bodies, whose duty should be to co-ordinate and control the standards of these examinations, to secure the interchangeability of certificates, and to consider and as far as possible to adjust the relations of the examining bodies and their spheres of external action.

Although absolute identity of standard between examinations conducted by different bodies and in different places may be an impossible ideal, practical equivalence can probably be secured. Further, more than one combination of subjects may be held to represent a good general education. It will be the duty of the Central Board to see that a sufficient minimum standard is maintained in each subject, so that certificates including these subjects, wherever given, may possess a generally recognised and interchangeable value, and further, that these certificates represent in each case a good general education.

(7) That the Board of Education should constitute this Central Board, as soon as, in their opinion, a sufficient number of recognised examining bodies have signified their willingness to be represented thereupon, and should take all steps that may be necessary to procure the acceptance of the certificates by the professional bodies.

(8) That since an examination held with the co-operation of the school in which a scholar has been taught is more likely to lead to a just estimate of the knowledge which he possesses than one held entirely by an outside body, the examination should be conducted in each school by **external and internal examiners**, representing respectively the examining body and the school staff.

(9) That the course of the work pursued by a scholar during his school career should be recorded and reported on by his teachers, and that this **school record and report** should be available for reference in deciding his fitness or unfitness to obtain a certificate.

The suggestion here is that an examiner, in any case in which he desires to do so, should be able to judge of the character of a candidate's school career. The school records and reports need not be of uniform pattern. What is required is that such materials shall be accessible as will enable an examiner to judge whether the scholar's school career has been satisfactory or not. These materials will include, at the least, the curriculum of all the classes which the candidate has attended, a note of the time he spent in each, and periodical reports of his industry, regularity, and progress.

(10) That the headmaster or headmistress of the school should certify that the candidate has received instruction during the necessary period, and is in his or her opinion, fit to enter for the examination.

(11) That the external examiner or examiners should have control of the examination, and should have a veto on the passing of any candidate.

(12) That the papers should be set by the external examiner, after consultation with the internal examiner.

This consultation does not necessarily involve a series of personal interviews previous to the examination. Full information as to the courses of study pursued by the candidate would in the first instance be supplied to the examining body for the information of the external examiner. The books read, whether in English, classical or foreign literature, and the courses of history or geography studied, the practical work done, etc., would thus be reported. The internal examiner, also, would suggest series of questions, or indicate points upon which, in his opinion, questions, should be set; and in general the two examiners would correspond on the subject matter of the examination paper. The paper should, however, be finally made up by the external examiner on his own responsibility.

(13) That the allocation of work in reading and marking papers should be determined by the examining body, provided that papers which are near the minimum pass mark should be considered by both examiners.

(14) That oral and practical examinations should be conducted by the external and internal examiners acting in concert, who should, subject to section 11, jointly assess the marks for each candidate in this part of any examination.

(15) That in language examinations no special books should be prescribed, but that passages should be included from the books used in the school as well as unseen passages. That an oral examination should always be held in the case of modern languages.

(16) That there should be a **senior certificate** for pupils who have received not less than four years' instruction in a school or schools accepted for examination under section (5).

That there should be a **junior certificate** limited to pupils under 16 years of age who have received not less than three years' instruction in a school or schools accepted for examination under section (5).

With reference to the number of years of instruction required, it will be desirable at first to give some latitude to the Central Board.

(17) That no certificates for honours or marks for special distinction should be given, but that it should be open to the examiners to recommend the award of scholarships within a school or group of schools, when called upon to do so.

(18) That scholars who are in a school which, in the opinion of the Board of Education, is unable to conform to these regulations, might be allowed to enter for the examinations under special regulations approved by Central Board.

UNIVOCAL CURVES AND ALGEBRAIC CURVES ON A QUADRIC SURFACE.

§ 1. The projective theory of conics is based chiefly on the fact that a conic is a unicursal curve, and the simplest analytical method of treatment is to take for reference a triangle formed by two tangents and their chord of contact, so that with suitable multiples of distances for coordinates the equation of the conic may be written in the form

$$y^2 - xz = 0.$$

The coordinates of any point on it may be expressed in terms of a parameter u by the equations

$$x/u^2 = y/u = z,$$

and as u takes all values from $-\infty$ to $+\infty$ the point (x, y, z), or (u) as it will be called, describes the complete curve exactly once.

In order to form a mental picture of the curve we may put $z=1$ and imagine the lines $xy=0$ to be at right angles. The conic becomes a parabola, and the parametric representation is the usual one. The point (0) is the vertex and (∞) is the infinite end of the axis, and the points (u) and $(-u)$ are reflexions in the axis.

The parameters of two points (k_1) and (k_2) on the conic are the roots of the quadratic equation

$$u^2-(k_1+k_2)u+k_1k_2=0,$$

which shows that the equation of the chord joining them is

$$x-(k_1+k_2)y+k_1k_2z=0.$$

Making $k_1=k_2=k$, we find that the tangent at (k) is

$$x-2ky+k^2z=0.$$

If (x, y, z) is regarded as a given point, this equation is a quadratic in k whose two roots u and v are the parameters of the points of contact of the tangents from (x, y, z) to the conic and satisfy

$$x/uv=2y/(u+v)=z.$$

Hence u and v may be regarded as new coordinates of the point (x, y, z), and the equation of any curve S becomes, after substitution for x, y, z, a symmetric equation in u and v,

$$S(u, v)=0.$$

Conversely, every symmetric equation represents a curve, for it is expressible in terms of uv and $u+v$, and therefore in terms of x/z and y/z.

The new coordinates of any point are interchangeable, that is, the points (u, v) and (v, u) are the same. So also every point on S has two pairs of coordinates, for if $S(u, v)=0$, then $S(v, u)=0$.

§ 2. $S(u, v)$ has no symmetric factors, for if it had, the curve S would be reducible, and this is supposed not to be the case. But among all algebraic curves there is a special class for which $S(u, v)$ breaks into complementary factors $\phi(u, v)$, $\phi(v, u)$. Then it is sufficient to retain only one factor and write the equation of the curve in the form

$$\phi(u, v)=0.$$

Of the two pairs of coordinates belonging to any point of S, only one satisfies this equation, and is therefore distinguished from the other pair which satisfies $\phi(v, u)=0$. For this reason S is called a *univocal* curve* and is represented by a general irreducible unsymmetric equation in u and v.

The simplest univocal curve is given by

$$u-k=0,$$

representing the tangent at (k). The corresponding symmetric equation is

$$(u-k)(v-k)=0,$$

and by means of the equations

$$x/uv=2y/(u+v)=z,$$

this becomes the previously found tangent

$$x-2ky+k^2z=0.$$

The univocal curve $u-v=0$

is the conic itself, and we have

$$z^2(u-v)^2=4(y^2-xz)=C_2 \text{ say.}$$

The general linear equation $au+bv+c=0$

gives the conic $(a-b)^2(y^2-xz)=\{(a+b)y+cz\}^2$

having double contact with $C_2=0$.

* "Courbe univoque." See Humbert, *Journal de Math.* 1893, sér. 4, t. 9, p. 154.

§ 3. Let $\phi(u, v)$ be of degrees m in u and n in v, and suppose $m > n$; then we may write

$$z^m\{\phi(u, v)+\phi(v, u)\}=S_m(x, y, z),$$

and

$$z^{m-1}\{\phi(u, v)-\phi(v, u)\}=(u-v)S_{m-1}(x, y, z),$$

S_m and S_{m-1} being homogeneous polynomials of degrees indicated by their suffixes. On squaring and subtracting we get

$$4z^{2m}\phi(u, v)\phi(v, u)=S_m^2-C_2S_{m-1}^2,$$

showing that the univocal curve touches the conic $C_2=0$ at all common points, and further that *at all points of the curve, $\sqrt{C_2}$ has the value S_m/S_{m-1} which is rational in the point coordinates x, y, z.* It will be proved that this latter property is characteristic of univocal curves.

The points of contact with $C_2=0$ are given by the equation

$$\phi(u, u)=0$$

which has $m+n$ roots. Hence the curve $\phi=0$ is of order $m+n$, and $S_m^2-C_2S_{m-1}^2$ is divisible by z^{m-n}. The curve $S_m=0$ passes through the $m+n$ points of contact and $m-n$ times through the point (∞).

The univocal curve has a certain number of double points given by

$$\phi(u, v)=0, \quad \phi(v, u)=0, \quad u \pm v,$$

and these lie on both of the curves

$$S_m=0, \quad S_{m-1}=0.$$

§ 4. It is important to distinguish between curves which touch $C_2=0$ at all common points, and curves on which $\sqrt{C_2}$ is rational.

Let $S_{2n}=0$ be a curve of order $2n$ touching $C_2=0$ at $2n$ points. Through these a curve $S_n=0$ can be drawn, not meeting the conic elsewhere. Then $S_{2n}=0$ passes through all the intersections of $C_2=0$ and $S_n^2=0$, and accordingly,* there exists an identity of the form

$$S_{2n}=A_{2n-2}C_2+B_0S_n^2$$

which gives the general form of $S_{2n}=0$ satisfying the geometrical condition of $2n$-fold tangency. By similar reasoning, if $S_{2n+1}=0$ touches $C_2=0$ at the $2n$ points of intersection with $S_n=0$ and at the point of contact of the tangent $S_1=0$, then there exists an identity of the form

$$S_{2n+1}=A_{2n-1}C_2+S_1S_n^2.$$

In neither case has $\sqrt{C_2}$ a rational value on the curve, unless further conditions are satisfied.

Let $S=0$ be an irreducible curve at every point of which $\sqrt{C}$ has the rational value P/Q. Then the curve $P^2-Q^2C=0$ contains $S=0$ as one part ; in other words, $S(x, y, z)$ is a factor of P^2-Q^2C. Substitute

$$x/uv=2y/(u+v)=z$$

and then we have, as the necessary and sufficient condition, that S must be a factor of $\{P+Q(u-v)\}\{P+Q(v-u)\}$, S, P, Q being now symmetric functions of u and v. Now if S were a factor of $P+Q(u-v)$, it would also be a factor of $P+Q(v-u)$ and hence a common factor of P and Q, contrary to the hypothesis that $\sqrt{C}$ has the definite value P/Q when $S=0$. Further S has no symmetric factors since the curve $S=0$ is irreducible. Hence S must be the product of two unsymmetric irreducible factors $\phi(u, v)$, $\phi(v, u)$, one of which divides $P+Q(u-v)$ and the other $P+Q(v-u)$.

Hence the univocal curves are the only curves on which $\sqrt{C}$ is rational, and the general equation of such a curve, multiplied possibly by a power of z, is

$$S_m^2-CS_{m-1}^2=0.$$

§ 5. The conditions for an n-ic curve $S(x, y, z)=0$, having n points of contact with $y^2=xz$, to be univocal, are that $S(2uv, u+v, 2)$ must be the product of

* Nöther's theorem, *Math. Annalen.* Bd. 6, p. 351, Bd. 52, p. 593.

two factors $\phi(u, v)$, $\phi(v, u)$. No conditions are required for $n=1$ or 2, for the general equation of a tangent line

$$x-2ky+k^2z=0$$

gives, on being multiplied by z,

$$y^2-xz=(y-kz)^2;$$

and again the general equation of a conic having double contact is

$$y^2-xz=(lx+my+nz)^2.$$

In each of these cases it is evident that $\sqrt{y^2-xz}$ has a rational value. In terms of u and v, the equation of the last curve is

$$luv+pu+qv+n=0$$

where $2p=m+1$, $2q=m-1$, and the univocal conic is the reciprocal with respect to $y^2=xz$ of the envelope of chords joining corresponding points of two projective rows on $y^2=xz$.

A single condition must be satisfied in order that a cubic curve having triple contact may be univocal. The general equation of a cubic curve touching $y^2=xz$ at $u=0$, 1, ∞ is

$$(ax+2by+cz)(y^2-xz)=y^2(x-2y+z),$$

but $\sqrt{y^2-xz}$ is not rational on it unless

$$b=\sqrt{ac}+\sqrt{a}+\sqrt{c}.$$

§ 6. The theory of univocal curves is intimately connected with the theory of algebraic curves on a quadric. Let the conic $y^2=xz$ be the section of the quadric by the plane $t=0$. Take the fourth corner of the tetrahedron of reference to be the pole of this plane, then the equation of the quadric may be written

$$t^2=y^2-xz.$$

It will be convenient to imagine the point $x=y=z=0$ at infinity in a direction perpendicular to the plane $t=0$, and to regard t as the distance of any point from this plane.

Any algebraic curve on the surface is partially given by its projection

$$f(x, y, z)=0$$

on the plane $t=0$. If f is a *general* polynomial in its arguments, this equation represents a cylinder cutting the quadric in an irreducible curve which is its own reflexion in the plane $t=0$; that is, the curve cuts each generator of the cylinder in two points equidistant from $t=0$. From the point of view of geometry on the quadric, this is a special property not possessed by the general curve; in fact, a general curve together with its reflexion in $t=0$ form the complete intersection with a cylinder, and the problem is to find under what conditions the intersection is reducible.

The equation of any surface which contains a curve K but not its reflexion must involve an odd power of t and hence can be reduced by means of the equation of the quadric to an equation of the form

$$P(x, y, z)-tQ(x, y, z)=0$$

representing a *monoid* surface.* This shows that the projection of K is a univocal curve, for at every point of it $\sqrt{y^2-xz}$ has the rational value P/Q. The reflexion of K lies on the monoid

$$P(x, y, z)+tQ(x, y, z)=0.$$

§ 7. The chief difficulty in the study of algebraical curves on a given surface is that more than one additional equation is required, for the general curve is not the complete intersection of any two surfaces, but must be defined as the part common to several. In the case of a quadric we are able, by means of univocal curves, to represent any curve on the surface by a single equation.

* Cayley. *Coll. Papers*, vol. V., p. 7.

The parametric representation of the surface is given by

$$\frac{x}{uv}=\frac{2y}{u+v}=\frac{z}{1}=\frac{2t}{u-v},$$

so that the points (u, v) and (v, u), which in the plane were identical, are separated on the surface and are reflexions of each other in $t=0$.

By what has been shown, we have proved the fundamental theorem : *the equation of any curve on the surface is $\phi(u, v)=0$, where ϕ is a general polynomial in its arguments.*

The parametric curves $u=k$, $v=l$, are the generators

$$\left.\begin{aligned} y+t&=kz \\ y-t&=k^{-1}x \end{aligned}\right\} \text{ and } \left.\begin{aligned} y-t&=lz \\ y+t&=l^{-1}x \end{aligned}\right\}$$

respectively, intersecting in the point (k, l). If $\phi(u, v)$ is of degrees m in u and n in v, it represents a curve of order $m+n$ cutting each generator $u=$const. in n points and each generator $v=$const. in m points.

Suppose $m=n$; a typical term in $\phi(u, v)$ is $u^r v^s$, and if $r>s$ we have

$$z^m u^r v^s = z^{m-r}(y+t)^{r-s}x^s,$$

but if $r<s$ we have $\qquad z^m u^r v^s = z^{m-s}(y-t)^{s-r}x^r.$

In this way the equation $\qquad z^m\phi(u, v)=0$

becomes that of a surface of order m cutting the quadric in the $2m$-ic curve $\phi(u, v)=0$. Hence *every equation of the same degree in u and v represents a complete intersection.* From this follows Halphen's theorem* that any curve on a quadric together with a certain number of generators is a complete intersection ; for if $m>n$ the function $\phi(u, v)(v-k_1)\ldots(v-k_{m-n})$ is of the same degree m in u and in v, showing that the curve $\phi=0$ and any $m-n$ generators of the system $v=$const. lie on an m-ic surface.

R. W. H. T. Hudson.

MATHEMATICAL NOTES.

149. [A. 1. b.] *Note on some inequalities connected with the expressions*

$$w=\frac{\Sigma a\Sigma(a-b)(a-c)}{\Sigma a(a-b)(a-c)}, \qquad n=\frac{a+b+c}{a-b+c},$$

where a, b, c are three positive quantities in order of magnitude.

(i) n, w both lie between 1 and ∞.

(ii) n lies between w and 3 ; proved by showing that $n-w$ and $n-3$ are of opposite signs.

(iii) w lies between n and $\frac{1}{4}(n^2+3)$; proved by assuming n constant while w varies, or otherwise.

(iv) n lies between w and $(4w-3)^{\frac{1}{2}}$; proved from (iii), etc.

Hence the following series is in order of magnitude, the first term being ∞ or 1 according as the series is in descending or ascending order :

$$\infty \text{ or } 1,\ \tfrac{1}{4}(n^2+3),\ w,\ n,\ (4w-3)^{\frac{1}{2}},\ 3.$$

w and n are equal if either of them is equal to 1, 3, or ∞.

If $b+c=a'$, $c+a=b'$, $a+b=c'$, then a', b', c' are sides of a triangle ; and conversely, if a', b', c' are sides of a triangle in order of magnitude we can put $a'=b+c$, $b'=c+a$, $c'=a+b$. Substituting for a, b, c in terms of a', b', c' in the above result, and making some slight changes, and finally dropping the dashes, we obtain the following :

If a, b, c are sides of a triangle in order of magnitude, and w, n stand for

* *Bull. de la Soc. math. de France*, vol. I., p. 19.

the same expressions as above, then the following series is in order of magnitude (commencing with ∞ or 2 according as it descends or ascends)

$$\infty \text{ or } 2,\ n+\frac{(n-2)(n-3)}{n-1},\ w,\ n,\ \frac{2(w+6)^{\frac{1}{2}}}{(w+6)^{\frac{1}{2}}-(w-2)^{\frac{1}{2}}},\ 3.$$

w and n both lie between 2 and ∞, and are equal if either of them is equal to 2, 3, or ∞. F. S. MACAULAY.

150. [K. 20. f.] *Napier's Rule of Circular Parts.*

Let ABC be a spherical triangle right-angled at C. Then another triangle having the same circular parts may be found as follows.

(1) Take a triangle symmetrically equal to ABC and place the two together with their sides AC in contact. They will form an isosceles triangle whose sides are $2a$, c, c, whose angles are $2A$, B, B and whose altitude is b.

(2) Take the polar of this triangle. Its sides are $\pi-2A$, $\pi-B$, $\pi-B$ and its angles $\pi-2a$, $\pi-c$, $\pi-c$, and its altitude is the supplement of the altitude of the previous triangle, *i.e.* $\pi-b$ (as can easily be shown).

(3) Take the colunar triangle of the last triangle. Its sides are $\pi-2A$, B, B, and its angles $\pi-2a$, c, c, and its altitude b.

(4) Split this triangle into two right-angled triangles. The angles of these will be c, $\frac{1}{2}\pi-a$, $\frac{1}{2}\pi$, and the sides opposite them b, $\frac{1}{2}\pi-A$, B.

If $A'B'C'$ denote this new triangle, we have therefore

$$a'=b,\ b'=\tfrac{1}{2}\pi-A,\ \tfrac{1}{2}\pi-A'=\tfrac{1}{2}\pi-c,\ \tfrac{1}{2}\pi-c'=\tfrac{1}{2}\pi-B \text{ and } \tfrac{1}{2}\pi-B'=a.$$

The triangle $A'B'C'$ has therefore the same circular parts as ABC, moved one place round.

In the penultimate step of this process we had an isosceles triangle formed of $A'B'C'$ and a symmetrically equal triangle with their sides a' in contact. By placing two such triangles with their sides b' in contact and repeating the processes, we obtain a third triangle having the same circular parts moved one place further round, and so on.

This method is not so elegant as the geometric one given in text-books on spherical trigonometry, but I would say that it is easier to remember, as I never find it possible to demonstrate the geometric method to a class without referring to the text-book. G. H. BRYAN.

151. [P. 3. b.] *Note on Successive Inversion.*

[The Apollonian circles of a triangle ABC are the three loci $b.PB=c.PC$, $c.PC=a.PA$, and $a.PA=b.PB$. They have for diameters the segments of the sides between the feet of the internal and external bisectors of opposite angles. They cointersect in two real points, inverse to the circum-circle, lying on the line joining the circum-centre and symmedian point. The pedal triangles of these points are equilateral.]

Inverting the property that the medians of an equilateral triangle cointersect at angles of 120°, it follows that the A-circles of a triangle are coaxal and cut each other at angles of 120°.

If any point P be taken on the sides of an equilateral triangle whose centroid is O, the concentric circle of radius OP will intersect the sides in five other points, symmetrically situated two on each side. No matter what their number or what their order, successive reflections of P with respect to the medians give these five points and no more.

Inverting; no matter what their number or what their order, successive inversions of a point P with respect to the A-circles of the triangle ABC give five new points and no more.

The formal proof of the following property presents no difficulty and need only be stated; the pedal triangle of any point is similar to the pedal

triangle of the inverse of that point with respect to either of the A-circles (M'Clelland, Chap. VI., Ex. 25, p. 148). It depends on the fact that the sides of the pedal triangles of P are proportional to $a.PA$, $b.PB$, and $c.PC$.

Thus there are generally eleven points that have their pedal triangles similar to the pedal triangle of a given point P. They are the five points obtained by successive inversions of P with respect to the A-circles; the inverse Q of P with respect to the circum-circle; and the five points obtained by similarly inverting Q.

R. F. DAVIS.

[The above note arises from the discussion of a question suggested by the Rev. J. J. Milne; and assistance is gratefully acknowledged from Mr. C. E. Youngman and Mr. C. E. M'Vicker.]

152. [V. a.] *Echo from the Examination Room.*

"In the common system of logarithms the mantissa is the same for all numbers having the same magnificent digits."

REVIEWS.

Mechanics. By JOHN COX, Macdonald Professor of Physics in M'Gill University, Montreal; formerly Fellow of Trinity College, Cambridge. Cambridge University Press, 1904. Pp. xiv, 332. Price 9s.

Too often a school-book belongs to the class of books which are no books, but here excellent paper, large type, and beautiful diagrams and portraits give pleasure to the eye; nor does the author shrink from a well turned sentence or even an occasional joke.

Throughout the book may be seen the influence of Mach, to whom the work is dedicated, and to whom full acknowledgment is made in the preface. The author follows the historical development of the subject; indeed, as he states in his preface, the first book, "The Winning of the Principles," is to a great extent an abridgment of Mach's earlier chapters. In book II. the mathematical statement of the principles is taken up, followed in book III. by application to various problems, while book IV., on the elements of Rigid Dynamics, completes the work. This book, and indeed the whole work, is especially suited for the student of physics, to whom, even at an early stage in his career, a clear view of the outlines of dynamics of rotation is essential. It comprises a discussion of the compound and ballistic pendulum, flywheel, torsion balance, and Cavendish's experiment.

The author's attitude towards certain details of methods of instruction which have formed the subject of recent discussions here, may be described as, on the whole, conservative. Thus no special stress is laid on graphical methods. A tendency prevailed for a long period to ignore the just claims of graphical methods in elementary work, claims justified by this fact alone that, treated graphically, any one length or angle is as easily dealt with as another. Recently a tendency has been visible in certain quarters to lay undue stress on graphic methods almost to the exclusion of methods of calculation. Correspondence in *Nature* [May-June, 1904] has indicated divergence of opinion among experienced teachers as to the proper place of graphical work in elementary mechanics. It must certainly be acknowledged that many beginners are quite as ready to confuse the tension of a string with its length when attempting a graphical solution, as they ever are to mingle forces and moments when proceeding by the way of analysis.

Perhaps it would be worth while to enact that the "Frame diagram and Force diagram should always be entirely separate and should be drawn in different colours." No doubt, in dealing with an object such as "a weight of 10 pounds kept in position by a horizontal strut 3 feet long hinged to a wall, and by a string 5 feet long attached to a point in the wall," to draw a separate triangle of forces, is to many persons utterly revolting. However, well meant attempts to utilise the frame diagram, or part of it, as a force diagram are responsible for some queer blunders. Let the beginner have the pleasure and profit of

inventing such labour-saving devices for himself. When he invents them he may safely use them

The more advanced graphical methods for dealing with coplanar non-concurrent forces certainly prove difficult to many students, partly because a certain power of viewing the question as a whole and deciding on a plan of action before putting pencil to paper is necessary. But after all, the process of writing down equations on no definite plan does not succeed in analysis so often as students sometimes appear to believe.

The author's attempt to clear up the difficulties inherent in the notion of 'mass' is not entirely successful. If an angel made the attempt his efforts would probably be criticised.

Professor Fitzgerald (*Scientific Papers*, edited by Professor Larmor, p. 452) observed "of all those dynamical quantities of which students are generally expected to form clear and distinct ideas without any actual experience of the things themselves, the most abstruse and the one about which the most metaphysical statements are made is 'quantity of matter' or 'mass.'"

One can avoid using the word at all (Professor Greenhill), or, with Professor Love, one may define the frame with reference to which accelerations are measured, and then define the mass ratio of A and B as the reciprocal of the ratio of the accelerations determined in A and B by their mutual action. There is nothing if one adopts the former course to prevent the observation being made that a book would, we cannot help believing, remain a book at the distance of the moon: or the conclusion being drawn that weight is not a characteristic property of a single body. But a middle course, in which we define mass and erect our structure of mechanics upon the definition in a manner which is at once unexceptionable and yet acceptable to the beginner, is one which I should be very glad to find, but which I do not think the author has achieved.

Another burning topic which for the most part the author leaves on one side is that of "practical" questions. He does not share the delusion that a question about a rod 12 inches long, supported at its ends, and carrying a weight of 5 ounces at its middle point, is converted into a practical one by simply expanding the rod into a beam 12 feet long, carrying 5 tons. The practical part of the matter would relate to the mode of supporting the ends, so as to ensure vertica reactions and an adequate amount of bearing surface.

Such details are outside the scope of the author's design, though judicious allusions to experiments and practice are frequent. A practical moral might, however, perhaps with advantage, have been pointed out in connection with the formula

$$T = T_0 e^{\mu\phi}$$

for the tension of a rope round a rough post. From a practical point of view, the feature of the formula is that it makes the ratio $T : T_0$ independent of the diameter of the post. This is because we have assumed the rope to be perfectly flexible. Real rope cannot be treated as perfectly flexible unless the diameter of the post is sufficiently large.

Most of us know the formula as one of which it is by no means easy to furnish an elementary proof, sound in every detail, and a student who found the proof given on page 200 unconvincing would have the author's sympathy (p. 201) and mine.

In his preface the author refers to his book as an experiment. It is one which, it may be hoped, has to-day a chance of obtaining the success which it deserves, a success which, not long ago, would have been impossible for a book which "aims at no particular examinations." No book on elementary mechanics would appear better suited for use where the historical, mathematical, and physical sides of the subject are deemed equally worthy of cultivation. C. S. JACKSON.

The Tutorial Algebra. (Advanced Course.) By W. BRIGGS and G. H. BRYAN. Pp. viii, 612. 6s. 6d. 1903. (Tutorial Press.)

We are glad to see that this excellent volume is now in its second edition. There are no alterations in the present edition, but it will be found well up to date, and we may mention incidentally that it contains a chapter of nearly thirty pages on the graphical representation of functions and continuity of functions.

The Tutorial Statics. By W. Briggs and G. H. Bryan. Third edition. Pp. vii, 366. 3s. 6d. **The Tutorial Dynamics.** By the same Authors. Second edition. Pp. viii, 416. 3s. 6d. 1903. (Tutorial Press.)

These books seem to have made their mark, and are now re-issued with some alterations and additions. In the absence of any text-book on the market dealing with the elements of Mechanics from a practical standpoint, and as an introduction to what is a very difficult subject, these will serve as well as many of their rivals and better than most.

Examen des différentes Méthodes employées pour résoudre les Problèmes de Géometrie. By G. Lamé. Pp. 124. 1818. (Courcier.) Reprinted by A. Hermann, Paris.

Mr. A. Hermann is one of those philanthropists among publishers who do not object to the risk involved in issuing a reprint of a famous book, even when they may not see an immediate return for the outlay. He is eclectic in his choice, and the volumes he reprints have, it is needless to say, some other quality than that of mere rarity. A special interest attaches to the present volume in that although it was written at an early stage in the development of the author—when he was still an engineering student in the "Corps Royal des Mines"—yet scattered here and there throughout the book are to be found remarks which shew how strongly marked even at that age was the author's critical faculty. For this reason it is essentially a book which a teacher may read with pleasure and advantage. The price is five francs.

Mathematical Papers of the late George Green. Edited by (the late) N. M. Ferrers. Facsimile reprint. Pp. xii, 336. 20 frcs. 1903. (Hermann.)

George Green entered upon his short but brilliant University career at the age of forty. It was but natural that at such an age he would feel the course of training necessary for a high place in the Tripos to be exceedingly irksome. And this was the case, but his mathematical ability was so unmistakable that he came out fourth Wrangler, and two years later was elected to a Fellowship of his College. Unfortunately for the scientific world, after yet another two years he was touched by the hand of Death, the great leveller of all human aspirations. We can but feel a poignant regret when a life so full of promise is taken from among us, whether it be that of a man of mature age such as George Green or of younger types as Abel, who died at 26, or the even still younger Galois, who perished in a duel at 21.

The best known of his writings is an essay "On the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," which was published by public subscription at Nottingham five years before he went to Cambridge. Here we find for the first time the word "potential" used to denote the function representing "the sum of all the particles" in the system divided by their respective distances from a given point.

Here, too, we find the famous formula in triple integrals which bears his name, the parent of analogous functions in magnetism and hydrodynamics. In applying the potential function to the discussion of the theories of magnetism and electricity the author proves many propositions that were later to be experimentally and independently demonstrated by Faraday. Nothing is more striking than the way in which theorems discovered by one writer are re-discovered years after by another. For instance the theorem on p. 64 referring to the density of the electric fluid on a certain conducting body was re-discovered twelve years later by three men to whom Green's work was unknown:—Chasles, Gauss, and W. Thomson. In the paper on the Reflexion and Refraction of Sound at the surface of separation of any two fluid media when a disturbance is propagated from one medium to the other, Green improves on the laborious methods of Poisson and reaches different results. When the velocity of transmisison of a wave in one medium is greater than in the other, by sufficiently increasing the angle of incidence in the first medium we can cause the refracted wave in the second to disappear. If we assume the vibration of the incident wave to follow a law such as that of the cycloidal pendulum, then, as the angle of incidence increases, the intensity of the reflected wave remains unaltered, while the phase of the vibration gradually changes. Duhem has pointed out a curious coincidence in connection with the equations of Green in this discussion.

Four geometers were pursuing their researches independently and simultaneously in France, England, Germany, and Ireland respectively, and each hailed "cette forme d'intégrale de l'équation des petits mouvements" as the key to the effects of total reflection in Acoustics and Optics. They were Cauchy, Green, Neumann, and MacCullagh.

Essai sur Hyperespace, Le Temps, La Matière, L'Energie. By M. BOUCHER. Pp. 204. 1903. 2 fr. 50. (Felix Alcan.)

This interesting little brochure deals with subjects on the borderland of mathematics and philosophy. The object of the author is to give as clearly as may be a popular account of what the words matter, energy, time and space should connote, but the greater part of the book is devoted to the consideration of the different forms of non-Euclidean space. The author is familiar with the current philosophical controversial literature on the subject, but we do not see from the list of authors referred to that he has embarked on any course of general reading such as would qualify him to engage the especial sympathy of mathematicians. He certainly quotes from M. Barbarin's little summary in the *Collection Scientia*, and from Poincaré's essay on non-Euclidean geometry, published in 1891. His acquaintance with English writings seems to be limited to Hinton's charming romances, to one of which he alludes as "a new ora of thocight" (p. 167) and more than once he draws the attention of the reader to the chapter on hyperspace in Mr. Rouse Ball's "Mathematical Recreations and Problems." An appendix of thirty pages gives an abstract of Mr. Stringham's paper in vol. iii of the "American (!) Journal of Mathematics." A scanty knowledge of English is no excuse for lack of familiarity with Mr. Russell's Foundations of Geometry, which has been translated into French, and as there is no mention of Klein, or Riemann, nor of Peano and Pieri, we may fairly conclude that the author has derived his knowledge from the philosophical articles and works in his list ranging from Plato to Lechalas, Renouvier, and Freycinet. M. Boucher would appear to be one of those to whom the idea of hyperspace is a fascinating topic, inasmuch as it gives us glimpses, as it were, of the possibility of other conditions of existence than those with which we are acquainted. The *motif* of his work is clearly the conviction that "la Raison se met d'accord avec cette appréhension presque unanime de l'humanité qui veut la distinguer de la matière." From this point of view the book is well worth reading. As we have said, his knowledge of current controversial literature is extensive, and he has compiled a very interesting account of the modern philosophical attitude towards the idea of non-Euclidean space. Some of our readers may care to send to the *Gazette* their opinion of the following "proof" by Galileo that we cannot have a series composed of an infinite number of terms, a proof which apparently receives the approval of M. Boucher. Among the first ten natural numbers there are only three squares; among the first hundred there are only ten; among the first thousand there are only thirty-one; and so on. So that the number of squares is always less than the number of natural numbers. But an infinite series of numbers contains not only the square of each number, but the square of its square, etc. "La contradiction étant manifeste, il est évident que la suite des nombres ne peut être prolongée à l'infini."

The Teacher's Blackboard Arithmetic. Part I. By "TACT." Pp. 107. 1s. 6d. 1904. (Blackie.)

Although the scope of this little book is rather below that of the work done in most secondary schools, being limited to the first four rules, yet it is worth while calling attention to its merits as a carefully thought-out and systematic introduction to the ideas of number and the elementary operations. Some of the devices are quite ingenious and all are fully and clearly explained.

I Tre Problemi Classici degli Antichi. Problema Terzo:—Trisezione dell' Angolo. By PROF. BELLINO CARRARA. Pp. 60. 1904. (Fusi, Pavia.)

In this little pamphlet Prof. Carrara brings to a close his historical study of the three famous problems of antiquity. We note that although authors are quoted up to 1899, there is no mention of Klein's well-known "Vorträge über ausgewählte Fragen der Elementargeometrie." In the trisection by means of the conchoid and circle Viviani is forgotten. We might enquire of our readers

en passant if there is any construction for solving this problem by means of the *cissoid*, as is alleged in Chambers's *Encyclopædia* (cf. *Nature*, 1901, p. 400). For the benefit of our younger readers we venture to print the following note sent to Catalan by M.B. (Catalan, by the bye, referred to the fact that M.B. signed himself "*ancien Professeur de Mathématiques*" as a "*circonstance aggravante*").

To divide an angle AOB or any arc AB into three successive parts proportional to the numbers 3, 4, 5.

Take $OA = OB$. Complete the circle centre O, radius OA or OB, and draw the diameter AOC. Produce CB and OB to E and D respectively so that $BD = BE = OB$. Let EO cut the circle in F, DF cut the circle in G, and GE cut the circle in H. Then shall $BF : FH : HA = 3 : 4 : 5$.

An interesting section deals with various instruments for the practical solution of the problem, invented by Varignon, Fusinieri (spelled elsewhere Fusineri), Amadori, General Plebani, and—Good Arthur. With the name of the latter gentleman (apparently quoted in Enriques', (also spelled Euriques), "Questioni riguardanti la Geometria Elementare"), we admit, to our shame, we are not acquainted. Perhaps the name has been taken from an index, thus: "Good, Arthur." There are again many misprints in the names of authors and in the titles of books—a blot which marred, as we have previously pointed out, the value of the first two parts of this series. Taken as a whole, the three will be found full of interest and an adequate compilation dealing with the three classical problems which were the object of so much ingenious research and speculation in ancient times.

Einleitung in die Analytische Geometrie der höhern Algebraischen Kurven nach den Methoden von Jean Paul de Gua de Malves. By P. SAUERBECK. Pp. vi, 166. 1902. (Teubner.)

De Gua's name is best known to the mathematician as the author of the "analytical triangle." He was Professor of Philosophy at a time when philosophy had a wider connotation than it at present enjoys, for under the influence of continental ideas the word science has gradually come to cover the ground of what once was called natural philosophy or simply philosophy. According to Mr. Rouse Ball, De Gua published in 1740 a book on analytical geometry, applying it to find the tangents, asymptotes, and singular points of an algebraical curve, without the aid of the calculus, and also shewing how singular points and isolated loops are affected by conical projection. The present essay has been inspired by Brill and Nöther's "Über die Entwicklung der Theorie der Algebraische Funktionen in alterer und neuerer Zeit." Dr. Sauerbeck, animated by a pious wish to give to the younger generation a faithful picture of the activity of a mathematician of a former time, has presented in modern guise the developments of De Gua's analytical methods, and has carried them on in the treatment of curves of the kind usually studied in modern text-books. A good idea and well carried out.

The Story of Arithmetic. By SUSAN CUNNINGTON, with a preface by Prof. W. H. H. HUDSON. Pp. xiv, 239. 3s. 6d. 1904. (Swan, Sonnenschein.)

This useful little book will be found to contain a clearly written and succinct survey of what the authoress calls the alphabet of mathematical study. In addition to what we naturally would expect to find in such a volume, there are a few amusing and interesting chapters dealing with the mysticism and folk-lore of Arithmetic, the Arithmetic of Shakespeare, Arithmetic in modern literature, and so forth. To some of our readers the following extracts from Lewis Carroll's *Life and Letters* may be new. As requirements for the Mathematical School at Oxford he enumerates: "(A) A very large room for calculating Greatest Common Measure. To this a small one might be added for Least Common Multiple: this, however, might be dispensed with. (B) A piece of open ground for keeping roots and practising their extraction; it would be advisable to keep square roots by themselves as their corners are apt to damage others. (C) A room for reducing Fractions to their lowest terms. . . . (D) A large room, which might be darkened and fitted up with a magic lantern, for the purpose of exhibiting circulating decimals in the act of circulation. This might also contain cupboards fitted with glass doors for keeping the various Scales of

Notation." . . . So much for Arithmetic, but we cannot refrain from quoting one more passage: "(E) A narrow strip of ground railed off and carefully levelled, for investigating the properties of Asymptotes, and testing practically whether Parallel Lines meet or no: for this purpose it should reach, to use the expressive language of Euclid, 'ever so far.' This last process of 'continually producing the lines' may require centuries or more; but such a period, though long in the life of an individual, is as nothing in the life of the University." Miss Cunnington has done her work well, and this volume will not only give the reader much useful information, but a good deal of innocent enjoyment.

Experimental and Theoretical Course of Geometry. By A. T. WARREN. Pp. iv, 259. (Clarendon Press.)

Mr. Warren's excellent course has had more than a *succès d'estime*, for in a few months it has reached a second edition. The author has taken the opportunity of including all the propositions required according to recent changes of regulations for Pass examinations at the Universities. He has also introduced references to the corresponding articles in the Experimental course, so that the experimental course may be read *pari passu* with the present volume.

Constructive Geometry, being steps in the synthesis of ideas regarding the properties and relations of geometrical figures; arranged for a first year's course in science. By J. G. KERR, LL.D. Pp. 121. 1s. 6d. 1904. (Blackie.)

This is a well thought-out introduction to geometry, in which, according to the plan adopted in the teaching of to-day, geometrical notions are built up in the same way "as that followed in dealing with the elements of chemistry or physics." There is more effort than in most of the books we have seen to make the pupil reason about properties and to deduce fresh results. In the hands of a skilful teacher it should go far to produce the habits of enquiry and thought, without which real progress in the higher branches of the subject is more than doubtful.

New School Arithmetic. Part I. By C. PENDLEBURY, assisted by F. E. ROBINSON. Pp. viii, 206, xxi. 2s. 6d. 1904. (Bell.)

It is eighteen years since Mr. Pendlebury's Arithmetic was published and at once took a place among the most popular books in the market. The recommendations of the Committee of the Mathematical Association as to the teaching of the subject are the immediate cause of a revised issue of this text-book. Among the more striking features of this new edition is the prominence which is rightly given to the Metric system of Weights and Measures. The new method of multiplication and the Italian method of division are taught from the outset, as is the additive method in subtraction. The utility of rough approximations as an initial stage in the working of a problem is insisted on from the first. The decimalising of money is inculcated in the commercial sections of the book, and we are glad to see that the authors are not afraid of using an algebraic symbol when it is an advantage to do so. We need only say that Mr. Pendlebury has long secured his place, and we rather imagine he will keep it.

Longmans' Senior Arithmetic for Schools and Colleges. By T. F. G. DEXTER and A. H. GARLICK. Pp. ix, 554. 1904. (Longmans.)

We feel that this Arithmetic will be very popular in Training Colleges and similar institutions. Types of every imaginable kind of question are worked out, and but few difficulties are left for the student to ponder over, except in the sets of problems which follow the chapters or sections. The number of examples is appalling. There are plenty of questions on the theory of the subject, an example which might well be followed in other manuals. There is a chapter on logarithms, and the author also deals with elementary mensuration of the circle, cone, and pyramid. The following may be taken as an instance of the expositional power of the authors:—"*The detection of misplaced terms.* Students when working problems from an arithmetic, fail sometimes to get the right answer. Some then 'dodge' for the answer by 'trying it another way,' *i.e.* they change the order of their terms. There is no need to indulge in this reprehensible practice

if the following method of detection is mastered—Take the right answer from the book and the wrong answer; form a ratio or fraction of the two answers, placing the incorrect one in the numerator and the correct one in the denominator; then the fraction thus formed will be (1) a square quantity if one ratio is misplaced; (2) the product of two square quantities if two ratios are misplaced." No explanation of this fact is vouchsafed. We might now understand why many teachers prefer their students not to have access to the answers. But the book is full of good points.

Preliminary Geometry. By R. ROBERTS. Pp. 56. 1s. 1904. (Blackie.)

This is likely to prove a useful little introduction to geometrical principles. It consists of a series of carefully graduated questions on the elementary parts of the subject, with test papers at intervals. The author claims that he has steered a middle course between discouraging the pupil by making the exercises too difficult, and, on the other hand, avoiding all need for mental effort. The claim is very largely justified.

Association of Teachers of Mathematics in the Middle States and Maryland. Bulletin No. I. July, 1904. Pp. 56. (Printed by the Association, New York.)

In the first year of what we hope will prove to be a vigorous existence of this new Society the number of members on its roll has nearly reached 300. The greater part of the Bulletin consists of papers read before the Association during the years 1903-4. They are on the "laboratory method" of teaching Mathematics; the Syllabus for Geometry in the grades; suggestions on topics for investigation by the Association; and on the "genuine applications" of Algebra. The first thing that strikes the reader is the closeness with which the improvements effected in our methods of teaching in this country have been followed by our brethren across the pond. The name of Perry is as well known to them as to us. Strained as the conditions of modern life may be in the States, we find in one of these papers the lament that "the school year is too short in most of our widely patronized private and endowed schools by reason of the vacation plans of their well-to-do patrons, and the luxurious tendencies of the times, which are inimical to strenuous application." It is a pity that the writers of some of these papers do not pay a little more attention to the virtues of literary form. Occasionally we find an outburst of feeble flamboyancy, but as a rule the style is bald and tame. Behold this purple patch! "It is indeed a source of satisfaction to know that so many individuals have awakened from a comatose state of dull perceptions sufficiently to realise that algebra has applications in physical science! We hear the cry 'Eureka' from the shores of Lake Michigan. This is the fountain of youth to the teacher, a promised land to the children after generations have died in the wilderness of old-fashioned mathematical teaching." But apart from the form of the papers there is little to be said against their content. Enthusiasm inspires them, and it is an enthusiasm which is clearly being organised and controlled. It is interesting to note the nature of the problems to which the attention of the American teacher is directed in the paper on topics for investigation by the Association. "What should be the aim of mathematical teaching? How can the bookwork in algebra and geometry be reduced to a minimum? How much exercise work (*examples*) should be done? How much time is necessary for teaching algebra and geometry intelligently (!) to the average class? Is it necessary to give so much time to Arithmetic, and do the results of arithmetical teaching justify the present large expenditure of time? What should be the character of the examinations? What is the maximum number of students in a class which a teacher of mathematics can instruct properly? What should be the qualifications of a teacher of mathematics? Which are the most essential chapters of algebra, geometry, and trigonometry? What parts of book-study could be entirely omitted? Which are the best methods for leading students into original mathematical work? What kind of original work is most suitable for the average secondary student? Which chapters of secondary school mathematics have practical value? Will the introduction of the 'laboratory method' reform mathematical teaching?"

We wish to the new Association a long life and all the success that the importance of its aims demands.

Aufgaben aus der Niederen Geometrie. By I. ALEXANDROFF. Pp. v, 123. 1904. (Teubner.)

This is the German edition of a useful introduction to methods of attacking the solution of geometrical riders. It was translated from the Russian into French some years ago, and was at the time reviewed in these columns. We are glad to call attention to the German edition, as to the student who knows French and wishes to pick up German mathematical terminology, the two volumes may prove of considerable value, quite apart from the intrinsic merits of the book.

Lehrbuch der Differenzenrechnung. By D. SELIWANOFF. Pp. vi, 92. 1904. (Teubner.)

This is a handy little introduction to the Calculus of Finite Differences, simply and clearly written. The first part deals with introductory matter, interpolation, and the approximate values of certain integrals. The second treats of summation, the J. Bernouilli Function, and Euler's summation formula with its applications. The third is devoted to linear difference equations of the first order, and linear difference equations with constant coefficients—the whole being subject matter of Boole's first eleven chapters, but treated lightly and on a smaller scale.

Lehrbuch der Analytischen Geometrie. By O. FORT and O. SCHLOMILCH. Vol. I. **Analytische Geometrie der Ebene.** 7th edition. Revised by R. HEGER. 1904. (Teubner.)

This, the seventh edition of a book, the special characteristic of which is its clearness and simplicity of exposition, is revised and improved in various details by Professor Heger of the Polytechnic at Dresden. The second, third, and fourth editions were edited by O. Fort, and it is one-and-twenty years since the fifth edition was revised by Professor Heger. The ninth and tenth chapters deal with lines of a higher order than the second, with transcendental lines in general, and spirals and the usual roulettes in particular.

Méthodes de Résolution et de Discussion des Problèmes de Géométrie. By G. LEMAIRE. Pp. 224. 1904. (Vuibert et Nony.)

This is a good representative of a type of compilation of which we have but few instances in this country. It covers much the same ground as Petersen and Alexandroff, both of which are available in French. There is something to be said for the view that a boy taught, as until recently he has here been taught, will not necessarily take "in his stride" and without preparation in some detail, such a grasp of the different methods by which a problem may be attacked as he will after the study of such a volume as this. The names of the sections will sufficiently indicate the lines on which the book is constructed: Loci; method of the intersection of loci; determination of a line (envelopes, lines through a fixed point, lines parallel to a given direction); translation, rotation, symmetry, method of similar figures; figures similarly and similarly situated, inversion; transformation and division of figures. Many of the solutions of problems are accompanied by discussion in detail, most unusual with us, but a process especially commendable in the case of intelligent boys, who often find this method a source of much inspiration.

Advanced Course in Algebra. By W. WELLS. Pp. viii, 581. 6s. 6d. 1904. (Heath.)

The course that is covered in this volume is hardly what we should call "advanced" algebra in this country, and yet we are told it is intended to meet the needs of "Colleges and Scientific Schools of the highest rank." For the book contains no more than is to be found in Smith's *Treatise*, or Hall & Knight's *Higher Algebra*—not so much, indeed, for the subject is not treated with the same detail. It takes nearly four hundred pages to bring the student to the end of the chapter on the binomial theorem, and only one hundred and fifty pages are devoted to permutations and combinations, probability, continued fractions, summation of series, theory of numbers, determinants, and theory of equations. Simple, simultaneous, and quadratic equations take up one-fifth of the book. Graphs are well to the fore in dealing with equations generally, the theory of the irrational number, derivatives, multiple roots, Sturm's theorem, and discontinuous functions. The chapter on involution contains the binomial theorem for positive integral

exponents, proved by induction. The explanations are generally thoughtfully and clearly put, but now and then are overdone—as, for instance, in the case of the finding of the square and cube roots of an algebraical expression. We notice as an example of the author's reluctance to enter into detail, that he is content with instructing the reader to reject the negative value of n, where n is the number of terms in an A.P., without attempting to give any interpretation of the apparent anomaly. The following is the only definition of a function we can find: "A function of a number is any expression which contains the number. Thus the expression $2x^2-3ax+5a^2$ is a function of x." If it were worth defining at all it was worth doing better than this. But on the whole the book is sound, carefully considered, and equal to, if not better than, any American text-book we have seen. The questions at the ends of the chapters seem well selected, and the book itself is well printed and nicely bound.

The Collected Mathematical Papers of James Joseph Sylvester. Vol. I. (1837-1853). Pp. 650. 18s. net. (Cambridge Univ. Press.)

"As an artist delights in recalling the particular time and atmospheric effects under which he has composed a favourite sketch, so I hope to be excused putting on record that it was in listening to one of the magnificent choruses in the 'Israel in Egypt' that, unsought and unsolicited, like a ray of light, silently stole into my mind the idea (simple, but previously unperceived) of the equivalence of the Sturmian residues to the denominator series formed by the reverse convergents. The idea was just what was wanting, the key-note to the due and perfect evolution of the theory." "It is, as I take pleasure in repeating, to a hint from Mr. Cayley, who habitually discourses pearls and rubies, that I am indebted for the precious and pregnant observation. . . . A distinguished mathematical friend in Paris communicated to me with great admiration Professor Hesse's result overnight. I ventured to affirm that to one conversant with the calculus of forms, the problem could offer no manner of difficulty. An hour's quiet reflection in bed the following morning or morning after sufficed to disclose to me the true principle of the solution. . . . Guided by an instinctive sense of the beautiful and fitting, in a happy moment I have succeeded in grasping this much-wished-for representation, with which I propose now and forever to take my farewell of this long and deeply cogitated theorem. A morning or two after the enquiry reached me, in a walk before breakfast by the side of the ornamental water in St. James's Park (a time and place by no means, according to my experience, unfavourable to the aspirations of the analytic muse) I had the satisfaction of falling on the rather piquant demonstration above given. . . ."

The above, taken at random, extracts from this stately tribute to the genius of Sylvester, the first volume of which lies before us, effectively illustrate certain characteristics of one whose name is written in letters of gold on the roll of English mathematicians. And although they are taken from papers which were written before he had reached the plenitude of his powers, the same characteristics stand out in deeper relief as time goes on, and will be as dominant in the last of the volumes of his collected papers as they are in the first. They exhibit in the most striking manner, occurring as they do in papers dealing with the most abstruse forms of mathematical thought, Sylvester's intense enthusiasm for his subject, and the fine spirit of loyalty and admiration for the intellectual powers of his great collaborateur in the development of the theory of invariants by which the writer was ever animated. It is rare indeed to come across notes so entirely, so innocently personal in the pages of a mathematical memoir.

The present volume contains 68 papers, varying in length from one or two to 160 pages. The longest is entitled "On the Theory of the Syzygetic Relations of two Rational Integral Functions, comprising an application to the Theory of Sturm's Functions. . . ." It also contains his dialytic method of elimination, the Essay on Canonical Forms, and some of the earlier of the investigations in the theory of Invariants. It is most interesting to trace the steps in the development of the latter theory, in which Sylvester played so important a part. "The theory of Invariants," says MacMahon, "sprang into existence under the strong hand of Cayley, but that it emerged finally a complete work of art, for the admiration of future generations of mathematicians, was largely owing to the flashes of inspiration with which Sylvester's intellect illuminated it." For thirty

years it was the main object of mathematical research both here and on the continent, a fact to which the names of Cayley, Sylvester, Hermite, Aronhold, Clebsch, and Brioschi bear sufficient testimony.

Our younger readers may wish to know more about this remarkable personality, and to them the following references may be useful, although they do not pretend to be complete: *The Eagle*, vol. xix, pp. 596-605 (with portrait); *Proc. London Math. Soc.*, vol. xxviii., pp. 581-586 (written by J. J. Walker); *Bull. of the American Math. Soc.*, 1897, pp. 299-309 (written by Fabian Franklin, an old pupil at the Johns Hopkins University); Nature Series of Scientific Worthies, *Nature*, vol. xxxix. (written by Cayley, with portrait); obituary notice in *Nature*, 1897 (written by Major MacMahon). Obituary notices also appeared in the *American Journal of Mathematics*, which he was instrumental in founding, and in *Science*, the latter being from the pen of one of his pupils, Prof. Halsted. There is a short biographical notice in the new edition of the *Encyclopædia Britannica*, quite unworthy of so brilliant a mathematician, and ending with a needlessly ungenerous sneer at Sylvester's poetical capacity.

SOLUTIONS.

497. [R. 4. c.] *Two uniform rods AB, AC of weights W, W' are rigidly joined at A, and AB is placed on a rough peg: shew that there is, in general, a length $> \frac{\mu W' p}{W+W'}$ on AB, such that if the peg touch AB anywhere within this space the system will be in equilibrium, p being the length of the perpendicular from C on AB and μ the coefficient of friction.* C.

Solution by R. F. MUIRHEAD.

Let B' and C' be the mid-points of AB, AC, and G the centre of gravity of the system, so that G lies in $B'C'$ and $B'G : GC' = W' : W$.

Let GM and $C'N'$ be perpendiculars let fall on AB, so that $C'N' = p/2$.

We have $$GM : C'N' = B'G : B'C' = W' : W' + W;$$

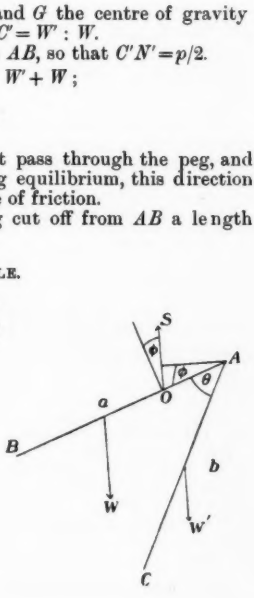
$$\therefore\ GM = \frac{p}{2} \times \frac{W'}{W'+W}.$$

For equilibrium, the vertical through G must pass through the peg, and give the direction of the reaction. For limiting equilibrium, this direction must make an angle with GM equal to the angle of friction.

Hence the two limiting positions of the peg cut off from AB a length $= 2\mu \times GM = \mu p W'/(W'+W)$.

Solution by A. W. POOLE.

(1) Let rods be just about to slip down. Also $\tan\phi = \mu$. $\theta_2 = \angle BAC$. Action at peg must be vertical, for W and W' are vertical.



We have $$S = W + W',$$

and taking moments about A

$$Sx\cos\phi = W\frac{a}{2}\cos\phi + W'\frac{b}{2}\cos(\theta+\phi),$$

where x is distance of peg from A;

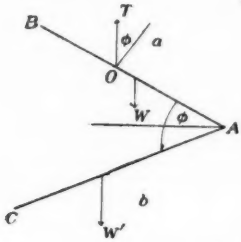
$$\therefore\ x = \frac{Wa\cos\phi + W'b\cos(\theta+\phi)}{2(W+W')\cos\phi}$$

$$= \frac{Wa + W'b[\cos\theta - \mu\sin\theta]}{2(W+W')}$$

(2) Let friction be limiting as before, the rods slipping in the other direction.

$$T = W + W',$$

and taking moments about A



$$Tx' \cos\phi = W\frac{a}{2}\cos\phi + W'\frac{b}{2}\cos(\theta - \phi);$$

$$\therefore\ x' = \frac{Wa + W'b[\cos\theta + \mu\sin\theta]}{2(W + W')},$$

where x' is distance of peg from A;

$$\therefore\ x' - x = \frac{2W'b\mu\sin\theta}{2(W + W')}$$

$$= \frac{W'\mu p}{W + W'};$$

$\therefore$ for equilibrium the peg must be in this length.

ERRATUM.

P. 3; line 13. For 2 read ι.

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The American Journal of Mathematics. Edited by F. MORLEY. Vol. XXVI., No. 2. April, 1904. $1.50. (Kegan, Paul.)

[On Nullsystems in Space of Five Dimensions. *J. Biesland.* On the Forms of Unicursal Quintic Curves. *P. Field.* Determination of Algebraic Curves whose Polar Conics are Parabolae. *E. Kasner.* On certain Conics connected with Trinodal Quartics. *A. B. Bassett.* A Geometric Proposition. *E. Lasker.* Congruences of Tangents to a Surface and Derived Congruences. *L. P. Eisenhart.*]

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